Boundary Value Problems & Sturm-Liouville Theory Differential Equations X. Du

- Referred to as BVPs
- BVPs are differential equations with a set of constraints known as boundary conditions, which specify the behavior of the solution at the boundaries.
- Numerical methods. The main numerical method of solving a BVP is the **shooting method**, which reduces the BVP to an IVP by guessing the initial condition which solves the BVP.
- **Homogeneous BVP**s are homogeneous differential equations with boundary conditions of 0.
  - Example: $y''+2x^2y^2=0$, $y(0)=0$, $y(1)=0$.
- **Sturm-Liouville equations**
  - A significant subclass of BVPs
  - Take the form $\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+q(x)y=-\lambda w(x)y$
    - $[p(x)y']'+q(x)y=-\lambda w(x)y$
  - Assume the following on an interval $I=[a,b]$:
    - $p(x)>0$, $w(x)>0$
    - $p(x)$, $p'(x)$, $q(x)$, $w(x)$ are continuous
  - Furthermore:
    - $\alpha_1 y(a)+\alpha_2 y'(a)=0$ $\alpha_1{}^2+\alpha_2{}^2>0$
    - $\beta_1 y(b)+\beta_2 y'(b)=0$ $\beta_1{}^2+\beta_2{}^2>0$ (coefficients are real)
- **Sturm-Liouville theory** refers to the analysis of Sturm-Liouville equations. Its fundamental tenets are:
  - The eigenvalues of this equation are real and its ordered sequence approaches infinity (i.e. $\lambda_1<\lambda_2<\lambda_3<\ldots<\lambda_n<\ldots\to\infty$)
  - Each λ_n has a unique corresponding eigenfunction y_n such that y_n has exactly $n-1$ roots on the open interval (a,b). y_n is called the nth fundamental solution
  - $\int_a^b y_n(x)y_m(x)w(x)dx=\begin{cases}0 & m\neq n\\ 1 & m=n\end{cases}$ Orthogonal and normalized
    - The set of eigenfunctions is orthonormal.